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The preventive role of Snell's law in mode conversion from Z- to whistler-mode waves in an inhomogeneous magnetoplasma with a low density

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Abstract

Electromagnetic waves with different modes, such as Z-, whistler-, LO- and RX- modes are found in different regions of the Earth magnetosphere and the magnetosphere of other planets. Since whistler-mode waves influence the behavior of the magnetosphere, and they are used as experimental tools to investigate the upper atmosphere, they are important. On the other hand, the mode conversion process can be considered as one of the processes of generating electromagnetic waves that can occur under certain conditions. Usually, propagation waves in an inhomogeneous plasma are a necessary, but not a sufficient condition for a mode conversion process. Snell's law has an important role in the mode conversion process. Although, this law lets a mode conversion occur from Z- to LO-mode waves, in a case from Z- mode to whistler mode waves, it plays a preventive role. The aim of this paper is to demonstrate the preventive role of Snell's law in a mode conversion from Z- to Whistler-mode waves in an inhomogeneous magnetoplasma with a low density. We used the dispersion relation in the magnetoplasma with a low density and for an oblique wave normal angle. By applying the Snell's law, we showed that with the propagation of the Z-mode waves in an inhomogeneous plasma, there is not any matching point between Z- and Whistler mode waves, and for any wave normal angle always an evanescent layer exists between the two modes. In this case, Snell's law prevents the mode conversion from occurring. It also prevents the transfer of energy from one to another mode waves.

Keywords: whistler, Snell's law, inhomogeneous plasma, mode conversion, magnetosphere

1 Introduction

In a magnetoplasma with a law density, $\omega_p < \omega_c$, where ω_p and ω_c are angular plasma frequency and cyclotron angular frequency, respectively, the whistler mode is a kind of electromagnetic wave, with an upper cutoff angular frequency at the plasma frequency, so that $\omega_w < \omega_p$, where, ω_w , is angular frequency of the whistler mode. Whistler mode waves have an R-hand polarization and they are found in all regions of Earth's

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magnetosphere, ionosphere and the magnetosphere of other planets. Also, these waves may originate in sources residing outside the magnetosphere, such as lightning (Storey, 1953).

Usually, whistler mode waves propagate along the magnetic field line. Whistler mode waves and their interactions with energetic particles is the most important subject. They also contribute to the pitch angle scattering and precipitation of the radiation belt electrons (Inan and Bell, 1977; Inan et al., 2003). A discussion of wave energy into and out of a duct is presented by James (1972).

On the other hand, another electromagnetic wave mode that usually is found in the plasmosphere and magnetosphere of the Earth is the Z-mode wave (Oya, 1971; James, 1979, 1991; Jones, 1977). In a magnetized plasma with a low density, the Z-mode waves have an upper oblique resonance, ω_{ZI}, and a cutoff, ω_Z, which are given by:

$$\omega_Z = \frac{\omega_c}{2}\left[-1+\left(1+4\frac{\omega_p^2}{\omega_c^2}\right)^{1/2}\right], \tag{1}$$

$$\omega_{ZI} = \frac{1}{\sqrt{2}}\left[\omega_{uh}^2+\left(\omega_{uh}^4-4\omega_c^2\omega_p^2\cos^2\theta\right)^{1/2}\right]^{1/2}, \tag{2}$$

where, ω_{uh}, θ, are the upper hybrid resonance and wave normal angle, respectively. Z-mode waves under certain conditions can be converted to LO- mode and escape to free space (Oya, 1971; Jones, 1988: Kalaee et al., 2009, 2010). In magnetized plasma the refractive index is determined by characteristics of a wave, such as wave frequency, wave normal angle as well as by parameters of plasma such as the density of electron and magnitude of external magnetic field. Therefore, in an inhomogeneous plasma, where these parameters vary as a function of position, the local refractive index is a function of position (Stix, 1992). The mode conversion, identified as a change in the propagation modes of plasma waves, is one of the generation mechanisms of radio emissions occurring in an inhomogeneous plasma, so that inhomogeneity is a necessary condition for mode conversion. However, it is not a sufficient condition (Kalaee and Katoh, 2014a).

Principally, two mode waves, when matched together at the point where the local plasma frequency becomes equal to the incident wave frequency while at this point the wave vector becomes parallel to the magnetic field line at the same time $(\theta = 0)$, with the exception of these conditions, mode conversion can occur just by tunneling effect (Kalaee and Katoh, 2014b). The two consequences of wave propagation in an inhomogeneous plasma can be changed by the wave vector, $\mathbf{k}$, and the wave normal angle θ, of the incident wave. In the case of mode conversion from Z-mode to LO-mode waves, it is possible that the two modes match together, since Snell's law provides this possibility. Therefore, this law that causes the parallel component of $\mathbf{k}$ vector remains constant during the wave propagation in an inhomogeneous plasma, has an important role in matching of the two modes. This law permits mode conversion to occur Z- to LO-mode waves, but, the role of Snell's law is different in the case of mode conversion from Z- mode to whistler mode waves. The aim of this paper is to show the role of Snell's law in a case of mode conversion from a Z- mode to a whistler mode wave. In order to show this role, we used the dispersion relation in the magnetoplasma with a low density and for an oblique wave normal angle. By applying the Snell's law, we showed that with the propagation of Z-mode waves in an inhomogeneous plasma, there is not any matching point between the Z- and whistler mode waves. In this case, Snell's law prevents the mode conversion from occurring and it prevents the transfer of energy from one to another mode waves.

2 Dispersion relations for the Z- and whistler mode waves

A homogeneous equation for the electric field can be expressed in the following from,

$$\mathbf{n}\times\left(\mathbf{n}\times\tilde{\mathbf{E}}\right)+\overleftrightarrow{\mathbf{K}}\cdot\tilde{\mathbf{E}}=0\,, \tag{3}$$

where the matrix $\overleftrightarrow{\mathbf{K}}$ is the dielectric tensor, $\tilde{\mathbf{E}}$ is the Fourier transform of the electric field, and $\mathbf{n}$ is the refractive index. By assuming that the magnetic field is directed along the z-axis, and that the wave vector, $\mathbf{k}$, lies on the xz-plane; the dispersion relation of the plasma wave and the eigenmode Equation (3) is determined by considering the condition of the non-trivial solutions of the homogeneous equation,

$$\begin{vmatrix} S-n^2\cos^2\theta & -iD & n^2\cos\theta\sin\theta \\ iD & S-n^2 & \theta \\ n^2\cos\theta\sin\theta & 0 & P-n^2\sin^2\theta \end{vmatrix}=0, \tag{3}$$

where,

$$S=\frac{R+L}{2},\ D=\frac{R-L}{2},\ P=1-\frac{\omega_p}{\omega} \tag{4}$$

and

$$R=1-\frac{\omega_p^2}{\omega^2}\left(\frac{\omega}{\omega+\omega_c}\right), \tag{5}$$

$$L=1-\frac{\omega_p^2}{\omega^2}\left(\frac{\omega}{\omega-\omega_c}\right). \tag{6}$$

We calculated dispersion relation for all mode waves, such as the RX-, LO-, Z- and whistler- mode waves in the typical low density plasma, with $\omega_p/\omega_c=0.5$, and for several wave normal angles, i.e. $\theta=0^\circ$, $\theta=5^\circ$, $\theta=25^\circ$, $\theta=45^\circ$, $\theta=90^\circ$. We also assumed that the external magnetic field was perpendicular to the electron density gradient. Figure 1 shows the results of the dispersion relation calculation in the wavenumber-frequency space.

Table 1. The parameters used in Figure 1; wave normal angle, upper hybrid resonance frequency, cutoff frequency and upper oblique resonance frequency.

θ°	ω_{UHR}/ω_c	ω_Z/ω_c	ω_{ZI}/ω_c
0. 0	1.118	0.207	1.0
5.0	1.118	0.207	1.011
25	1.118	0.207	1.028
45	1.118	0.207	1.068
90	1.118	0.207	1.118

In our calculation, all of frequencies were normalized by the cyclotron frequency. Table 1 shows the parameters that used in Figure 1 with the values of the upper oblique resonance, ω_{Zi}/ω_c, and the cutoff, ω_Z/ω_c.

3 Dispersion relations by considering Snell's law

For this case, assuming that the external magnetic field is perpendicular to the electron density gradient, the parallel component of the wave vector should be constant so that the wave propagates in an inhomogenous plasma environment. Under this obligation, we calculated dispersion relations for a certain frequency of the incident wave, $\omega / \omega_c = 2.03$ in an inhomogenous plasma.

Figure 2 shows the results of the dispersion relations for two typical examples of the Z-mode and LO-mode waves in the wave normal angle- plasma frequency space.

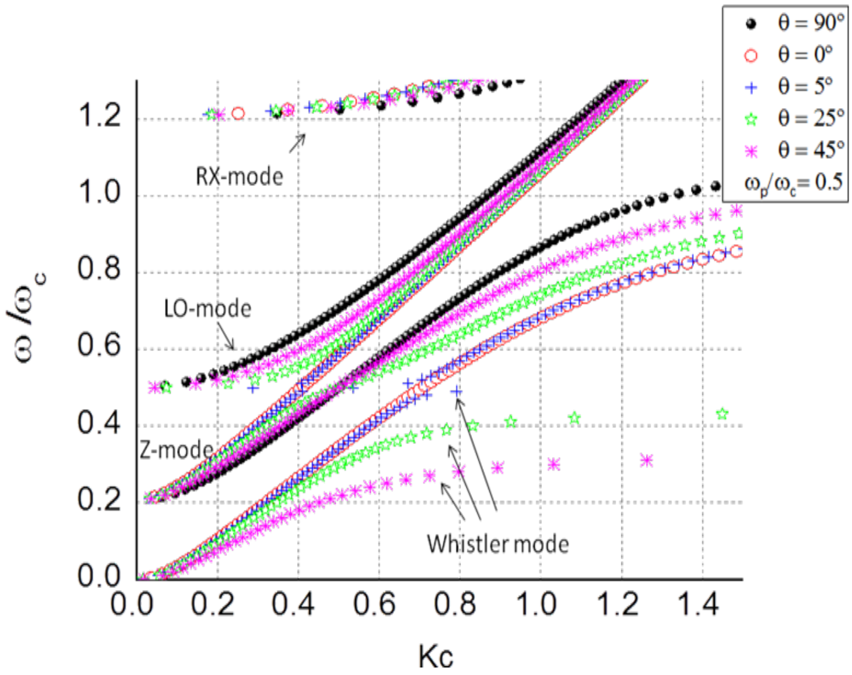

Figure 1. The ω-k diagram with the several wave normal angles for all mode waves in a magnetoplasma with $\omega_p / \omega_c = 0.5$.

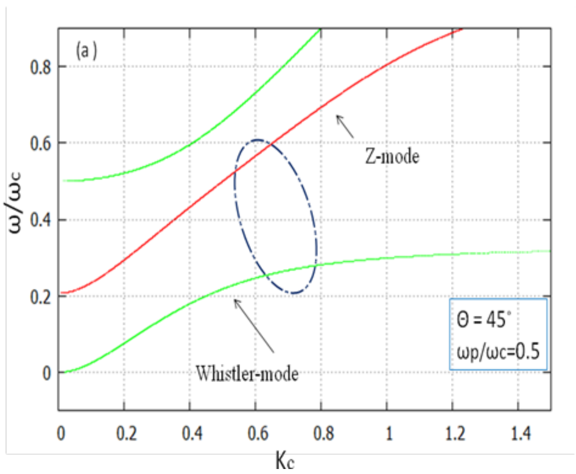

Figure 2. Variation of wave normal angle depends on the local plasma frequency by considering the Snell's law, for the case that $\omega / \omega_c = 2.03$. Under certain conditions, two mode waves can match (the green curve). The Snell's law lets the mode conversion occur from Z- to LO-mode waves.

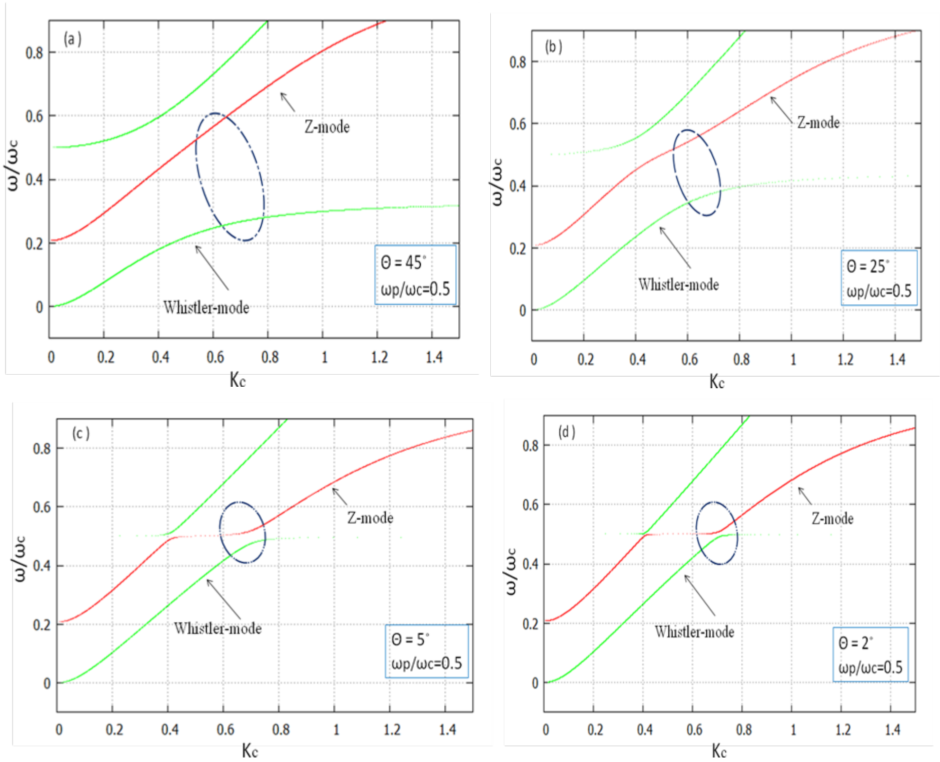

Figure 3. The ω-k diagram for whistler and Z- mode waves with $\omega_p / \omega_c = 0.5$, and (a) $\theta = 45^\circ$, (b) $\theta = 25^\circ$, (c) $\theta = 5^\circ$, (d) $\theta = 2^\circ$. Two mode waves approach as the wave normal angle decreases (see inside the circles).

The Figure indicates the variation of the wave normal angle with the local plasma frequency. As can be seen from the graph, under a certain condition, the two mode waves (LO and Z) can be matched together where the wave normal angle becomes zero (green curve). However, in general, the two mode waves are mismatched (the red curve). In the next section, we will address dispersion relations by considering the Snell's law for the Z- and whistler mode waves and discuss the results of calculations.

4 Results and discussion

In the previous section, by applying Snell's law for the Z-mode and LO mode waves, we showed that in general two mode waves were mismatched (the red curve), so that the mode conversion cannot occur, except by tunneling effect (Kalaee and Katoh, 2014). However, under a certain condition, the two mode waves can match and mode conversion can occur (green curve).

The condition for matching modes is satisfied when the parallel component of the incident wave refractive index, $n_{\|}$ is equal to the critical value, $n_{\|}^c$.

Therefore, for the Z-mode wave, it is possible to convert to LO-mode wave (Jones, 1976, 1980, 1988; Kalaee et al. 2009, 2010). As shown in Figure 2 the Snell's law lets the mode conversion occur from Z- to LO-mode waves if $n_{\|} = n_{\|}^c$.

Finally, we considered the dispersion relation for the Z- and whistler mode waves with a low-density plasma. We considered the propagation of the Z-mode in an inhomogeneous plasma by considering the Snell's law. Under this obligation, we numerically calculated the dispersion relations for a certain frequency of the incident wave, and for the entire wave normal angles of the incident wave. First, we calculate the dispersion relations for the Z-mode and the whistler mode waves with $\omega_p / \omega_c = 0.5$ and all of the wave normal angles, without the obligation of the Snell role. Figures 3(a-d) show the results for $\theta = 45^\circ$, $\theta = 25^\circ$, $\theta = 5^\circ$ and $\theta = 2^\circ$. As shown in Figure 3, the two modes (Z- and whistler) approach as the wave normal angle decreases (see inside the circles). Therefore, if we skip over the Snell's role, it may seem that the two mode waves can match as the wave normal angle approaches zero.

However, when the wave propagates in an inhomogenous plasma, the Snell's law has an important role so that the parallel competent of the incident wave vector should be constant during the propagation wave propagation and it should not be ignored. In the next step, we considered the Snell's law and calculated the dispersion relation with this obligation. For this purpose, we could use Figure 1 to obtain the parallel component of the *k* vector, for a certain frequency and wave normal angle of the whistler wave, e.g. at the $\omega_p / \omega_c = 0.5$, the initial parallel component of the *k* vectors are equal to 0.75 and 0.566 for $\theta = 25^\circ$, $\theta = 45^\circ$, and $\omega / \omega_c = 0.40$, $\omega / \omega_c = 0.28$, respectively.

In an inhomogenous plasma, the parallel competent of the incident wave vector should be constant during the propagation of the wave. We numerically calculated the dispersion relations for the angular frequency of the incident wave $\omega / \omega_c = 0.40$ and for the entire wave normal angles. Table 2 shows the values of parallel component of ***k*** vector for the whistler and Z-mode waves. Figure 4(a) shows the variations of the wave normal angle for the initial whistler mode wave during the wave propagation in an inhomogenous plasma by considering the Snell's law. For all of them, there is not any Z-mode wave with these parallel components of the *k* vector when the waves propagate in an inhomogenous plasma.

Table 2. The values of the parallel component of the *k* vector for the whistler and Z-mode waves, with the incident wave frequency of $0.4\omega_c$, during the wave propagation in an inhomogenous plasma.

$ck_{\|\|}$	**Z-mode**	**Whistler -mode**
0.45 0.47	√	nothing
0.49	√	nothing
0.50	√	nothing
0.51	√	nothing
0.53	√	nothing
0.55	nothing	√
0.60	nothing	√
0.75	nothing	√
1.10	nothing	√
	nothing	√

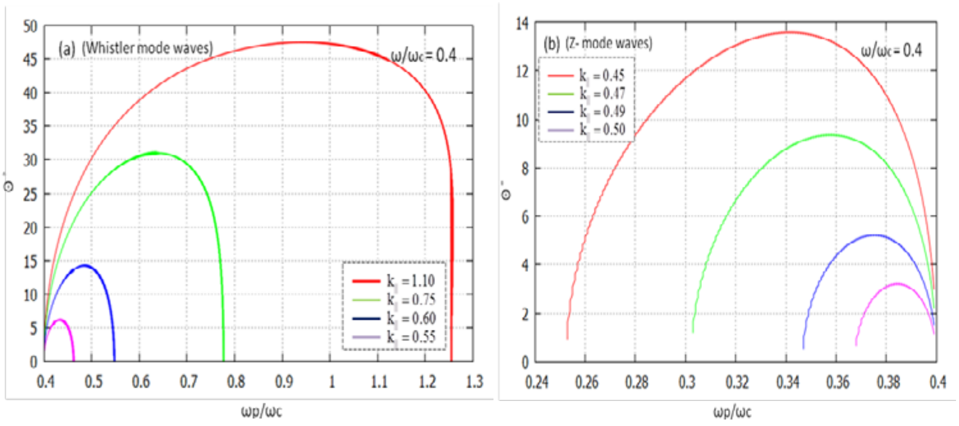

Figure 4. The variations of the wave normal angle for the initial waves during the wave propagation in an inhomogenous plasma, with different values of the parallel component of *k* vector and by considering the Snell's law; a) for the whistler mode waves and b) for the Z-mode waves. There is no common value between the two modes.

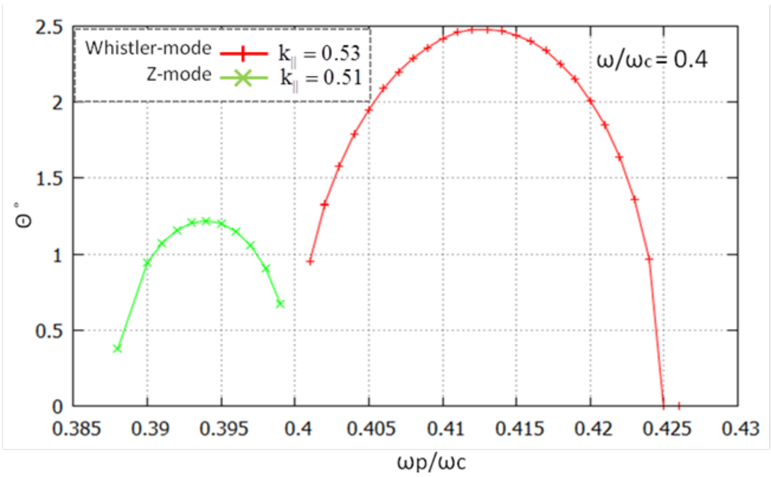

Figure 5. A comparison of the wave normal angle variations of the whistler mode with the wave normal angle variations of Z-mode waves, in the value of their bound, where the parallel components of the ***k*** vectors of the two waves are closer together. The two mode waves can approach in a small limited angle about zero, but never match with the zero angle at the point with the plasma frequency equal to the incident wave frequency.

In Figure 4(b), we show the variations of wave normal angle for the Z-mode during the wave propagation in an inhomogenous plasma by considering the Snell's law. Also, for all of them there was not any whistler-mode wave with these parallel components of *k* vector when the waves propagate in an inhomogenous plasma.

As can be seen from Table 2, for $(\omega / \omega_c = 0.40)$, the minimum value of the parallel component of the whistler mode wave is about 0.53 so that $k_{w\|} \geq 0.53$ and the maximum value of the parallel component of the Z- mode wave is about 0.51 so that $k_{z\|} \leq 0.51$. There is no common value between the two modes. As a result, two modes cannot match; the Snell's law prevents the mode conversion from occurring and it prevents the transfer of energy from one to another mode waves. In Figure 5, we compared the results of the wave normal angle variations of whistler mode with the wave normal angle variations of the Z-mode waves, in the value of their bound, where the parallel components of the ***k*** vectors of the two waves are closer together ($k_{w||} = 0.53$ and $k_{z||} = 0.51$). It can be seen from Figure 5 that the two mode waves can approach in a small limited angle about zero, but never match in a zero angle and at the point with the plasma frequency equal to the incident wave frequency.

Therefore, in such cases, the two modes waves can be converted together only by tunneling effect.
A summary of discussion is outlined below:

1. When the wave propagates in an inhomogenous plasma, for a certain value of the ***k*** vector parallel component, only the Z-mode wave exists, and for another value of the ***k*** vector parallel component only the whistler mode exist, $k_{w||} \neq k_{z||}$.
2. Two mode waves can approach in a small limited angle about zero, but never match in a zero angle and at the plasma frequency equal to the incident frequency.
3. In such cases, the two modes can be converted together just by tunneling effect.

5 Conclusions

In the present research, we studied the possibility of the mode conversion processes from the Z-mode to the whistler mode waves by considering the role of Snell's law. We used the dispersion relation in the magnetoplasma with a low density and for an oblique wave normal angle.

First, we discussed the conversion of the Z-mode to the LO-mode waves by applying the Snell's law. We showed under a certain condition, two mode waves can match and mode conversion can occur. The condition for matching modes is satisfied when the parallel component of the incident wave refractive index is equal to a critical value. Therefore, for the Z-mode wave, it is possible to convert to the LO-mode wave. The Snell's law lets the mode conversion occur from Z- to LO-mode waves if $n_{w||} = n_{z||}$.

Second, we considered the dispersion relation for the Z- and whistler mode waves with a low-density plasma. We numerically calculated the dispersion relations for a certain frequency of the incident wave, and for the entire wave normal angles of the incident wave.

We showed the variations of the wave normal angle for the Z-mode during the propagation of the wave in an inhomogenous plasma by considering the Snell's law. The results showed that for all of them there was not any whistler-mode wave with the parallel components of the *k* vector the same as the parallel components of the *k* vector for Z-mode wave, when the waves propagate in an inhomogenous plasma.

By applying the Snell's law, we showed that with the propagation of the Z-mode waves in an inhomogeneous plasma, there is not any matching point between Z- and whistler mode waves, and for any wave normal angle always an evanescent layer between two modes exists. As a result, in this case, Snell's law prevents the mode conversion from occurring, and it prevents the transfer of energy from one to another mode waves.

In some cases, two mode waves can approach in a small limited angle about zero, but never match with the zero angle at the point with the plasma frequency equal to the incident frequency. Therefore, in such cases, two modes can be converted together just by tunneling effect.

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