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IOSVRM method for earth gravity field estimation using GOCE satellite gravimetric observations and comparison with Tikhonov and TSVD methods

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Abstract
The ill-posed problems could be seen anywhere in our daily lives. An ill-posed problem is a problem that there are no uniqueness solutions (there is no solution or two or more solutions for the problem) or the solutions are unstable; i.e., an arbitrarily small error in the observation may lead to extremely large errors in the solutions. The main difficulty in solving ill-posed problems is instability of their solutions with respect to small variations of input data. A regularized estimation of an ill-posed problem is always biased; thus, it's worth obtaining the solution from different methods for reliable evaluation of the uncertainty in our estimation. The regularization methods, such as Tikhonov's method, are used to obtain stable solutions for solving ill-posed problems. In Tikhonov's regularization method, a scalar quantity is used as the stabilization parameter to solve ill-posed problems; whereas, In the Optimally Scaled Vector Regularization Method (OSVRM), a vector is used as the stabilization parameter. In this paper, a comparison has been made between the results of Tikhonov, TSVD, and OSVRM methods in terms of accuracy of the results for estimation of the earth gravity field from GOCE satellite data. The RMSE of the results of Tikhonov, EGM96 model, and the OSVRM method – that use a vector instead of scalar as regularization parameter – in the order of 10-5, 10-9, and 10-13, respectively. It is seen that the results obtained from the OSVRM method are much better compared to the Tikhonov method and EGM96 model for solving linear ill-posed problems. On the other hand, a significant improvement has been achieved in the stability and accuracy of numerical results for linear problems solution.

Keywords: Tikhonov stabilization method, regularization parameter, singular value decomposition, OSVRM method, GOCE satellite

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1 Introduction

According to Hadamard (1923), a problem is well-posed, if the following three conditions are met:

The question has an answer (existence of at least one solution for the problem).

The answer should be unique (single answer).

The solution changes continuously with changes in the data (Continuity).

The answer is a continuous function of the observations; In other words, small fluctuations in observations do not cause large fluctuations in the results (response stability).

The problem is said to be ill-posed, if at least one of the above conditions is not met. Therefore, we have to use regularization methods to solve the problem using basic information.

Suppose X and Y are the Normed spaces and the problem below is ill-posed:

$$\mathrm{V\,x} = \mathrm{y}, \qquad \mathrm{V}: \mathrm{X} \to Y \tag{1}$$

In classic numerical methods, the problem will be diverge, if the model was ill-posed. In other words, classical numerical methods such as Chulsky decomposition, LU decomposition, and QR decomposition do not have the ability to estimate a meaningful solution for Equation (1) in the discrete space; But by using different regularization methods, a stable answer can be obtained for the system of equations (Hansen, 1992).

Suppose that the f_δ is the value of function f by the approximation δ, such that $\|f - f_\delta\| \le \delta$. We are looking for an approximate answer to equation (1) from Q_δ:

$$Q_\delta = \{x \in X: \|Vx - f_\delta\| \le \delta\} \tag{2}$$

However, in the ill-posed problems, an arbitrary element $x_\delta \in Q_\delta$ cannot be considered as an approximate answer to equation (1). Since x_δ is not continuously dependent on f_δ, the expression $\|Vx - f_\delta\| \le \delta$ does not guarantee that x_δ is close to the desired solution for the system of equations. Therefore, in this case, we will have to use a priori information (known properties of the solution) to solve the problem and reach a stable solution. Using prior information to stabilize the answer is called the Tikhonov regularization method. In the case of a system of nonlinear equations, iterative methods can be used to arrive at a stable solution.

Basically, regularization methods find a unique stable solution for the problem by considering certain hypotheses and a known scalar parameter (the regularization parameter). Then, instead of finding the solution from the whole answer space, it must be focused on determining the regularization parameter. Therefore, in any regularization method, a unique solution for the ill-posed problem is not found directly, but a set of solutions is provided and it is expected that the real solution of the problem is a member of this set. Then, a unique solution would be selected from the set of answers by considering some specific hypotheses. Therefore, the discussion about regularization parameter is of special importance in regularization methods.

1 Tikhonov stabilization method

Consider the following linear equation system:

$$Vx = b_1 \quad , \quad V \in R^{n\times n} \quad , \quad x\,,y \in R^n \tag{3}$$

Matrix V is extremely ill-conditioned and has large single values. Such a system of linear equations is commonly referred to as ill-posed linear problems. An example of these equations is the Fredholm integral equations of the first type with a smooth kernel. Since the data values of matrix b were obtained through different observations, they were usually contaminated with a variety of measurement errors. In addition, they also sometimes have discretization errors. We represent the sum of these errors with $e \in Rn$.

Suppose $b \in R^n$ is the real and unknown value corresponding to b_1:

$$b_1 = b + e \tag{4}$$

Now consider the system of linear equations (3) with the unknown quantity of b:

$$Vx = b \tag{5}$$

Suppose that $\hat{x}$ has a solution for Equation

(5) (for example, the answer obtained by minimizing Euclidean norm). We want to obtain an approximation of $\hat{x}$ by calculating the approximate answer of the system of linear equations (3). Since the matrix, V is a bad-conditioned matrix and also, due to the error e on the right side of the equation (4), the direct solution of equation (3) does not give a meaningful solution for $\hat{x}$. In order to determine a meaningful approximation of $\hat{x}$, the system of linear equations (3) is replaced by another system that is close to the original system, but, less sensitive to data perturbations on the right. The solution of the new system is approximately assumed to be the value of $\hat{x}$. This replacement is usually interpreted as regularization.

There are several ways for stabilizing the problem. One of the most common stabilization methods is the Tikhonov Regularization Method. In this method, the system of linear equations (3) is replaced by the following minimization problem:

$$\min_{x \in R^n} \left\{ \|Vx - y\|^2 + \frac{1}{\mu} \|x\|^2 \right\} \tag{6}$$

In other words, in Tikhonov's Regularization method, in addition, to minimizing the norm of residuals, the solution norm $\|x\|$ is also minimized. In the above phrase, $\mu > 0$ is called the regularization parameter. The meaning of $\|.\|$ in phrase (6) is the Euclidean norm or $\|.\|_2$. The parameter μ determines the solution sensitivity of x_μ to the error e in the observations; also, how close the answer is to the real answer $\hat{x}$.

Generally, the Penalty term $\frac{1}{\mu}\|x\|^2$ can be replaced by the phrase $\frac{1}{\mu}\|Lx\|^2$. L is called the operator or regularization matrix. In Tikhonov's regularization method, a common option for matrix L is the identity matrix.

2 Determination of the regularization parameter

Methods for determining the regularization parameter are divided into three categories depending on the available information from the error norm ($\|e\|_2$):

Methods that are only based on a proper estimation of the error norm (initial parameter selection method)

Methods that depend on both observations and errors (secondary parameter selection method)

Methods that do not require information from the error norm and are based only on observations (error-independent method)

Since we usually do not have information about the error norm, the error-independent method is used to determine the regularization parameter. In this case, there are four methods to determine the regularization parameter:

Discrepancy principle (DP)

Generalized Cross Validation (GCV) method

L-curve method

The Flattest Slope method

For all regularization methods, the principle is to balance between the stability of the solution and the bias caused by the regularization parameter. Among the above methods, the L-curve method is the most suitable graphical tool for analyzing discrete ill-posed problems. The L-curve is a plot -for all valid regularization parameters- of the size of the regularized solution versus the size of the corresponding residual (Ardalan et al, 2008).

As mentioned earlier, the common point of Tikhonov and TSVD regularization methods is their dependence on the regularization parameter. The regularization parameter controls the amount of filtering applied by the regularization (Hansen & O'Leary, 1993). Therefore, the main point of these methods is to find the regularization parameter, so we can reach the final solution by removing or reduction of the noise, without losing any information. In

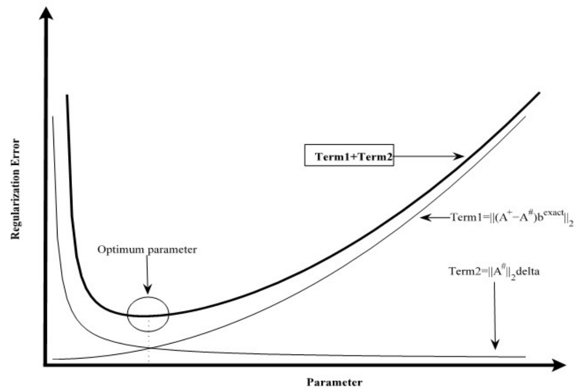

Figure 1. Optimal regularization parameter and the balance between bias and variance (Ardalan et al. 2008).

This Content, the L-curve is the most suitable graphical criterion for determining the regularization parameter that can be used in all regularization methods. In the L-curve method, the size of the norm of the stable solution (||x||) is plotted against the corresponding norm of residuals (||Ax − b||). In this method, the interpretation of the size depends on the application stabilization method. For example, L2 norm is suitable for the Tikhonov method, while norm L1 is suitable for the TSVD regularization method.

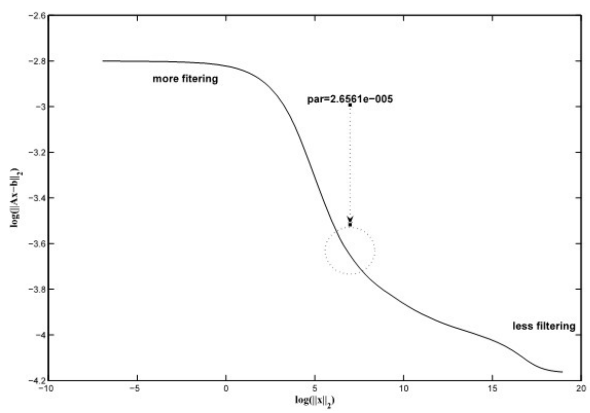

Figure 2. L-Curve [14].

Table 1. Types of regularization methods and their characteristics (Hansen & O'Leary , 1993; Hansen, 1992).

Method	Minimizes	Domain	Filter factors
Tikhonov	$\|r\|_2$	$\{x : \|x\|_2 \leq M(\lambda)\}$	$f_i = \frac{\sigma_i^2}{\lambda^2+\sigma_i^2}$
Truncated SVD	$\|r\|_{\underline{1}}$	$\{x : \|x\|_{\underline{1}} \leq M(\lambda)\}$	$f_i = 1, i = 1, \dots, k(M)$, $f_{k(M)+1} = \frac{\sigma_{k(M)+1}}{\alpha_{k(M)+1}}\left(M - \sum_{i=1}^{k(M)} \frac{\|\alpha_i\|}{\sigma_i}\right)$ $f_i = 0, i = k(M) + 2, \dots, n$
l_∞	$\|r\|_{\underline{\infty}}$	$\{x : \|x\|_{\underline{\infty}} \leq M(\lambda)\}$	$f_i = \min(1, \frac{M\sigma_i}{\|\alpha_i\|})$
LSQR	$\|r\|_2$	$\{x \in \mathcal{K}_k : \|x\|_2 \leq M(\lambda)\}$	No simple formula
Maximum entropy	$\|r\|_2$	$\{x \geq 0 : s(x) \geq M(\lambda)\}$	No simple formula

3 Truncated Singular Value Decomposition (TSVD)

This method is an extension of decomposing the eigenvalues of square operators, with the difference that in this method each matrix with any dimension can be multiplied by three matrices, one of which is diagonal and two other matrices are orthogonal or invertible.

In practical methods for solving the discrete problem in the form $Ax = y$ the effect of noise e must be minimized. Different regularization methods differ only in the way that they determine the noise-filtering function.

Matrix A can be decomposed by single value analysis (SVD) as follows:

$$A = U\sum V^2 \quad (7)$$

Where, $Um \times m = [u1 \dots um]$ and $Vn \times n = [v1 \dots vn]$ are orthogonal matrices.

In this relation, U and V are the left and right single vectors of matrix A that are orthonormal, respectively.

In general, the dimension of matrix U is m×m, whose columns are orthonormal eigenvectors of UU^T. Also, the columns of the matrix $V_{n\times n}$ are obtained from the orthonormal eigenvectors of V^TV matrix. The matrix $\Sigma_{m\times n}$ is a matrix whose diagonal elements are non-zero singular values of the VV^T or V^TV matrix.

To find the least squares solution of equation (3), it is used the single values decomposition of the matrix V by minimizing the following expression:

$$\min_{x\in R^n} Vx - y^2 \quad (8)$$

The solution is:

$$x_{LSQ} = \sum_{i=1}^{n} \frac{\alpha_i}{\sigma_i} \vartheta_i$$

(9)

Where: $\alpha_i = u_i^T y$.

The problem with using the least squares solution of x_{LSQ} is that the error will increase significantly in directions corresponding to the small singular values. Then, the information in directions corresponding to the larger singular values would be corrupted. Therefore, any practical method must involve the filter coefficients f_i in the estimation. The filtered solution is:

$$x_{filtered} = \sum_{i=1}^{n} f_i \frac{\alpha_i}{\sigma_i} \vartheta_i \quad (10)$$

Filter coefficients are usually assumed to be $0 \le f_i \le 1$. If all filter coefficients are selected as units, the answer will be equal to the solution of the least squares method. Different Regularization methods differ only in how the filter coefficients are selected. For example, in the Tikhonov method with the regularization parameter μ, the filter coefficients are:

$$f_i = \frac{\sigma_i^2}{\mu^2 + \sigma_i^2} \quad (11)$$

The answer obtained from the TSVD method is:

$$x_k = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i} v_i \quad (12)$$

In the generalized TGSVD method, the answer is as follows:

$$\hat{x}_{k,L} = \sum_{i=p-k+1}^{p} \frac{u_i^T b}{\sigma_i} v_i + \sum_{i=p+1}^{n} (u_i^T b) x_i \quad (13)$$

Table 1 lists the various regularization methods and the characteristics of each of them.

In general, if ρ(μ) would be the function that is minimized by the regularization method and the η(μ), the function corresponding to the stabilized solution x, then we can say the selection of regularization method is equal to the selection of a suitable pair of functions ρ and η. For example, in Tikhonov's method we have:

$$\rho(\mu) = Vx - y_2 \quad , \quad \eta(\mu) = Vx_2 \quad (14)$$

Also, the L-curve is a logarithmic graph of the values of the function η(μ) versus ρ(μ) to determine the optimal regularization parameter. In this case, the value of the regularization parameter is a point of the diagram that has the maximum curvature.

4 OSVRM method:

We want to solve the system of ill-posed linear equations:

$$Vx = y \quad (15)$$

In scientific problems, the accurate values of observations matrix y are seldom known; it's always accompanied by noise due to measurement error. Therefore, the solutions obtained by solving the problem at the ill-posed conditions may deviate greatly from the real answer. To avoid such a situation, the regularization methods are used and two common methods in this field, i.e. Tikhonov method and TSVD method were described in detail in previous sections. To solve the system of equations (1), two methods of Conjugate Gradient and Relaxed Steepest Descent Method (RSDM) are described, and then, the optimized OSVRM method is explained (Liu, 2012).

5-1 Conjugate Gradient and RSDM Methods

If we multiply equation (1) by V^T, we have:

$$V^T V x = V^T y \quad (16)$$

Assuming $C = V^T V$ and $V^T y = b$, we can rewrite the equation (16) as $\mathrm{C} = \mathrm{xb}$. For the iterative conjugate gradient method, the steps to solve the equation are summarized below.

1- With the known initial value x_0, the values r_0 and p_1 are calculated:

$$r_0 = b - C\,x_0 \qquad , \qquad p_1 = r_0$$

2- The iterative steps are performed for $k = 1, 2, \ldots$, as follows:

$$\alpha_k = \frac{\|\mathbf{r}_{k-1}\|^2}{\mathbf{p}_k^T \mathbf{C} \mathbf{p}_k},$$
$$\mathbf{x}_k = \mathbf{x}_{k-1} + \alpha_k \mathbf{p}_k,$$
$$\mathbf{r}_k = \mathbf{b} - \mathbf{C}\mathbf{x}_k,$$
$$\eta_k = \frac{\|\mathbf{r}_k\|^2}{\|\mathbf{r}_{k-1}\|^2},$$
$$\mathbf{p}_{k+1} = \mathbf{r}_k + \eta_k \mathbf{p}_k.$$

This iteration is continued until: $\|r_k\| < \varepsilon$
The steps of performing RSDM calculations (Liu, 2011) are presented to compare with the conjugate gradient method:
With the initial known value of x_0, the following steps are repeated for $k = 1, 2, \ldots$.

$$\mathbf{r}_k = \mathbf{C}\mathbf{x}_k - \mathbf{b},$$
$$\mathbf{x}_{k+1} = \mathbf{x}_k - (1-\gamma)\frac{\|\mathbf{r}_k\|^2}{\mathbf{r}_k^T \mathbf{C} \mathbf{r}_k}\mathbf{r}_k,$$

The iteration is continued until: $\|r_{k+1}\| < \varepsilon$. The parameter γ is a constant quantity: $0 \le \gamma < 1$.

$$\|\mathbf{r}_k\| < \in$$
$$\mathbf{r}_k = \mathbf{C}\mathbf{x}_k - \mathbf{b},$$
$$\mathbf{x}_{k+1} = \mathbf{x}_k - (1-\gamma)\frac{\|\mathbf{r}_k\|^2}{\mathbf{r}_k^T \mathbf{C} \mathbf{r}_k}\mathbf{r}_k,$$
$$\alpha_k = \frac{\|\mathbf{r}_{k-1}\|^2}{\mathbf{p}_k^T \mathbf{C} \mathbf{p}_k},$$
$$\mathbf{x}_k = \mathbf{x}_{k+1} + \alpha_k \mathbf{p}_k,$$
$$\mathbf{r}_k = \mathbf{b} - \mathbf{C}\mathbf{x}_k,$$
$$\eta_k = \frac{\|\mathbf{r}_k\|^2}{\|\mathbf{r}_{k-1}\|^2},$$
$$\mathbf{p}_{k+1} = \mathbf{r}_k + \eta_k \mathbf{p}_k.$$
$$\|\mathbf{r}_{k+1}\| < \in,$$
$$\mathbf{x}_{k+1}$$

4 OSVRM: A vector regularization method for solving ill-posed linear problem

Consider the matrix equation $VU = I_n$; In other words: $U = V - 1$. Since V is known based on Equation (1), so V^T is also known and we have x_0 as the initial input vector:

$$\mathbf{y}_0 = \mathbf{V}^T \mathbf{x}^0$$

then:

$$\mathbf{y}_0^T = (\mathbf{x}^0)^T \mathbf{V}$$

Multiplying the equation by U and considering $VU = I_n$, we have:

$$\mathbf{y}_0^T \mathbf{U} = (\mathbf{x}^0)^T$$

The above two equations form a system of over-determined linear equations to solve U, as follows:

$$\mathbf{BU} = \begin{bmatrix} \mathbf{I}_n \\ (\mathbf{x}^0)^T \end{bmatrix}, \qquad \mathbf{B} := \begin{bmatrix} \mathbf{V} \\ \mathbf{y}_0^T \end{bmatrix}$$

The dimensions of matrix B are: $(n+1) \times n$. Multiplying the above equation by U^T gives a matrix equation $n \times n$ to solve U:

$$\left[\mathbf{V}^T \mathbf{V} + \mathbf{y}_0 \mathbf{y}_0^T\right]\mathbf{U} = \mathbf{V}^T + \mathbf{y}_0 (\mathbf{x}^0)^T \qquad (17)$$

Multiplying the above linear operator by b_1 and assuming that Ub_1=x, we have:

$$\left[\mathbf{V}^T \mathbf{V} + \mathbf{y}_0 \mathbf{y}_0^T\right]\mathbf{x} = \mathbf{b} + (\mathbf{x}^0.\mathbf{b}_1)\mathbf{y}_0 \qquad (18)$$

It has been mathematically proved that the solutions of the above equation are the same as the solutions of Equation (1) (Liu et al, 2010). Of course, this equation also has similarities with Tikhonov's regularization equation, which is expressed by the following equation:

$$\left[\mathbf{V}^T \mathbf{V} + \alpha \mathbf{I}_n\right]\mathbf{x} = \mathbf{b} \qquad (19)$$

Tikhonov's regularization method disturbs the system of principal equations (1) by adding the regularization parameter α to the system of equations (19). In other words, on the right-hand side of equation (19), there is no sentence that compensates for the disorder caused by the addition sentence of the αx. This is why in Tikhonov's stabilization method if the regularization effect is over-emphasized, the accuracy of the resulting solutions may be greatly impaired.

On the other hand, the proposed vector regularization method does not disturb the system of original linear equations but mathematically transforms it into a new system of equations using the regularization vector $\mathbf{y}_0 = \mathbf{V}^T \mathbf{x}^0$. In other words, in this method, an additional sentence $(\mathbf{x}^0.\mathbf{b}_1)\mathbf{y}_0$ is added to the right-hand side of the equation to compensate for the regularization sentence $\mathbf{y}_0 \mathbf{y}_0^T \mathbf{x}$ on the left-hand side of the equation. Therefore, in Equation (18), both the stability and accuracy of the results are considered simultaneously.

According to Equation (18) and considering that $\mathbf{y}_0 = \mathbf{V}^T \mathbf{x}^0$, we have:

$$\mathbf{Ax} = \mathbf{b} + (\mathbf{x}^0.\mathbf{b}_1)\mathbf{V}^T \mathbf{x}^0 \qquad (20)$$

Where:

$$\mathbf{b} = \mathbf{V}^T \mathbf{b}_1$$

And

$$\mathbf{A} := \mathbf{V}^T \mathbf{V} + \mathbf{V}^T \mathbf{x}^0 (\mathbf{x}^0)^T \mathbf{V} \qquad (21)$$

In the above equations, if we could not select a suitable value for the regularization vector $\mathbf{x}^0$, these equations will still be ill-conditioned. Therefore, the problem is to find the appropriate value of the vector $\mathbf{x}^0$, so that the condition number of the matrix $\mathbf{A}$ is reduced as much as possible.

Theoretically, matrix $\mathbf{A}$ is in equilibrium if all its rows or columns have a norm, in which case the matrix is best-conditioned. According to the equilibrium matrix theory, the vector $\mathbf{x}^0$ can be chosen so that in equation (20), each row of the matrix of coefficients A has a Euclidean norm $R_0 > 0$:

$$\sum_{j=1}^{n} A_{1j}^2 = \ldots = \sum_{j=1}^{n} A_{nj}^2 = R_0^2 \qquad (22)$$

R_0 is a fixed scalar quantity that can be selected with the following equation:

$$R_0 \geq R_{\max} := \max_{i=1,\ldots,n} \sqrt{\sum_{k=1}^{n} C_{ik}^2} \qquad (23)$$

Where C_{ik} 's are the components of the $\mathbf{C} = \mathbf{V}^T \mathbf{V}$ matrix. Based on Equation (21), nonlinear algebraic equations (*NAEs*) can be concluded as follows:

$$F_i = \sum_{j=1}^{n} A_{nj}^2 - R_0^2 = \sum_{j=1}^{n}\sum_{k=1}^{n}(V_{ki}V_{kj} + \sum_{m=1}^{n} V_{ki}x_k^0 x_m^0 V_{mi})^2 - R_0^2 = 0, i = 1,\ldots,n \qquad (24)$$

The Jacobin B_{il} matrix of the above nonlinear equation system is written as follows:

$$B_{il} := \frac{\partial F_i}{\partial x_l^0} = \sum_{j=1}^{n}\sum_{k=1}^{n} 2A_{ij}[V_{li}x_k^0 V_{kj} + V_{ki}x_k^0 V_{lj}],\ i,l = 1,\dots,n \qquad (25)$$

In this case, an optimal iterative algorithm can be obtained to solve Equation

1) Select the value $\gamma\,(0<\gamma<1)$; then, with the initial value of x_0^0, the function $\mathbf{F}_0 = \mathbf{F}(\mathbf{x}_0^0)$ is computed.
2) For $k=1, 2, \dots$ the following iterative calculations are performed:

$$\mathbf{R}_k = \mathbf{B}_k^T \mathbf{F}_k,$$
$$\mathbf{C}_k = \mathbf{B}_k^T \mathbf{B}_k,$$
$$\mathbf{P}_k = \mathbf{R}_k - \frac{\|\mathbf{R}_k\|^2}{\mathbf{R}_k^T \mathbf{C}_k \mathbf{R}_k} \mathbf{C}_k \mathbf{R}_k,$$
$$\mathbf{v}_1^k = \mathbf{B}_k \mathbf{R}_k,$$
$$\mathbf{v}_2^k = \mathbf{B}_k \mathbf{P}_k,$$
$$\alpha_k = \frac{1}{1+\omega_k},$$
$$\beta_k = \frac{\omega_k}{1+\omega_k},$$
$$\mathbf{u}_k = \alpha_k \mathbf{R}_k + \beta_k \mathbf{P}_k,$$
$$\mathbf{v}_k = \alpha_k \mathbf{v}_1^k + \beta_k \mathbf{v}_2^k,$$
$$\mathbf{x}_{k+1}^0 = \mathbf{x}_k^0 - (1-\gamma)\frac{\mathbf{F}_k \cdot \mathbf{v}_k}{\|\mathbf{v}_k\|^2}\mathbf{u}_k$$

These steps are repeated until x_{k+1}^0 converges according to the criterion $\|F_{k+1}\| < \varepsilon$.
Once the x^0 vector is specified, the matrix of coefficients A (equation 21) is calculated. Then the conjugate gradient method can be used to solve the system of linear equations (20).

5 Results and Discussions

In this research, we retrieved the spherical harmonic coefficients from the GOCE satellite observations and compare the results with the coefficients of EGM96 model. Three different methods were tested in this paper: Tikhonov regularization method, TSVD method, and OSVRM method.

For this purpose, the whole globe was covered by a 1°×1° regular grid, and the center of each grid is considered as the estimation point. The gravity field recovery to a high degree and order coefficients (up to degree and order of 250) is possible for GOCE observations, but it requires a supercomputer. Thus, in this research, the coefficients of the gravity field were estimated to degree and order of 50.

The GOCE satellite data are the gradients of gravity acceleration (Kusche & Klees, 2002). Due to the large dimensions of the matrices, for estimation of coefficients, we used only the radial component of the gravity field (V_{zz}) from the GOCE observations, because in this case, the transformation of observations isn't necessary. The following tables show the results of different methods.

In Table 6, a comparison between the results obtained from three methods of regularization has been made. Table 6 shows the RMSE difference between the results of different regularization methods. The results were assessed for all coefficients up to degree and order of 50; however, to summarize the table, the RMSE criterion has been shown only for some degrees and orders of the estimated coefficients. In this table, the RMSE between results obtained from each regularization method (Tikhonov method, OSVRM method, and EGM96 model) and the coefficients resulting from GOCE observation is calculated for any degree and order.

In each column, the RMSE is given for the estimated coefficients up to the desired degree and order. The first column of table 6 shows the RMSE between the results of Tikhonov regularization method vs. the coefficients of GOCE data processing. The RMSE values are in the order of 10^{-5}. For the coefficients of the EGM model, the order of RMSE is about 10^{-9} which is much better than the Tikhonov method. In other words, Tikhonov method in this work hasn't been any significant effect on improving the results. On the other hand,

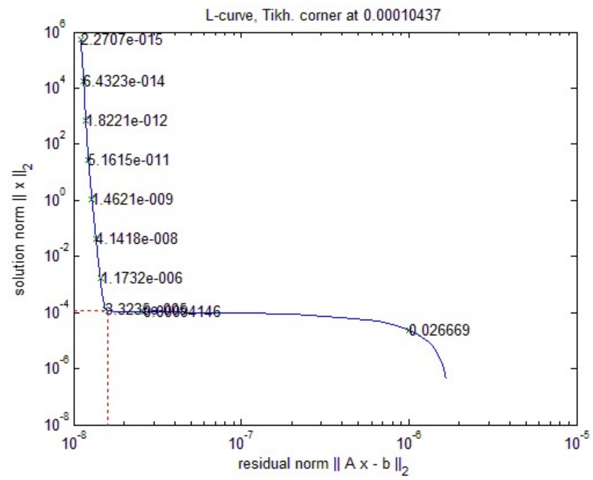

Figure 3. L-curve with Tikhonov.

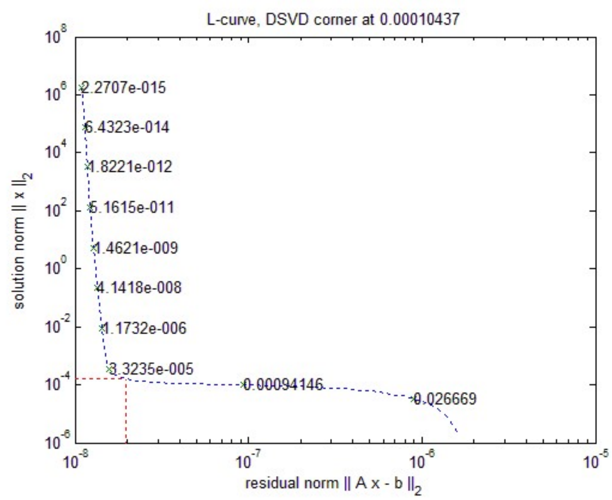

Figure 4. L-curve with DSVD.

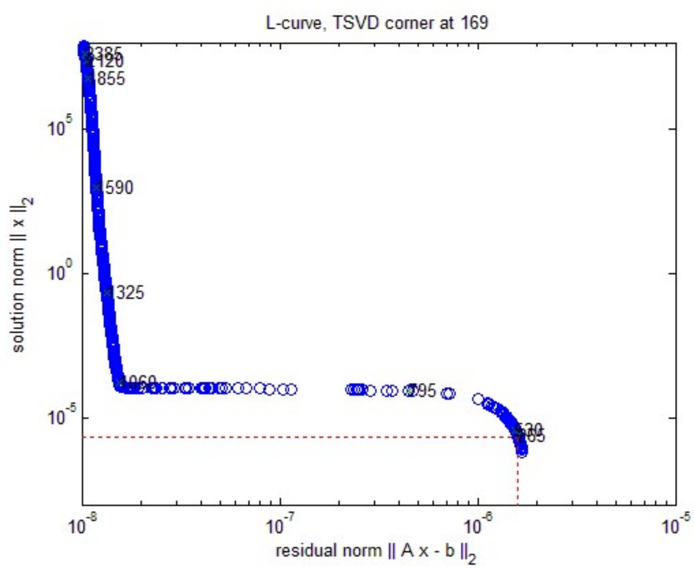

Figure 5. L-curve with TSVD.

Table 2. Results from GOCE observations and Tikhonov calculations.

		Tikhonov		GOCE	
n	m	C	S	C	S
2	0	-2.7419037772909884e-005	4.6236599212669352e-018	-4.841649667904e-04	0
2	1	4.8607119847842508e-007	-2.5010817100104957e-005	-2.525700823662e-10	1.395515029905e-09
2	2	5.0583554900263341e-006	-1.1841920980307721e-008	0.243936742466E-05	-.140024885157E-05
3	2	1.2957599005938219 0e-005	2.97417939394563520e-007	0.904802178874E-06	-.618981455772E-06
3	3	2.57547869753464580e-007	4.33512042169693330e-006	0.721276848787E-06	0.141440680154E-05
5	3	4.58102941812140310e-007	-6.4446625627251176e-006	-.451874314921E-06	-.214955260696E-06
5	4	4.59429448310091810e-007	5.71704476659061410e-007	-.295320219610E-06	0.497947374945E-07
5	5	-2.60294660349190950e-007	2.54851972563933850e-006	0.174795471385E-06	-.669368182338E-06
10	10	-1.89594975450603620e-007	-4.0043621051322067e-007	0.100439125877E-06	-.238750300838E-07
30	28	4.85407219765477410e-007	-3.0058539072562160e-007	-.584441578684E-08	-.809375913544E-08
30	29	4.11108072067448150e-007	-5.9562072871234156e-007	0.389521213237E-08	0.187969991193E-08
30	30	6.16027731493220000e-007	1.02940898081651560e-007	0.257229425698E-08	0.848735816571E-08
32	30	5.02937015895879430e-007	-1.1789644493924266e-007	-.690056829098E-08	0.132472373219E-08
35	34	4.15036347481664310e-007	8.39768128688313840e-008	-.798674881563E-09	0.276325910655E-08
50	49	5.52664991771392890e-008	2.22127671785326640e-007	0.215868304713E-08	-.475090336407E-08
50	50	-2.37399630073483230e-007	6.62684715626504590e-008	0.456424766124E-08	0.264773190543E-08

Table 3. Results from GOCE observations and dsvd calculations.

		dsvd		GOCE	
n	m	C	S	C	S
2	0	-2.649023587623442e-05	-2.088835579189738e-08	-4.841649667904e-04	0
2	1	-5.964828205257369e-08	-2.480122892375052e-05	-2.525700823662e-10	1.395515029905e-09
2	2	3.869884736605785e-06	5.707479486902128e-07	0.243936742466E-05	-.140024885157E-05
3	2	1.259857421249086e-05	8.661545926822130e-07	0.904802178874E-06	-.618981455772E-06
3	3	-1.006888615740551e-06	4.143079232967882e-06	0.721276848787E-06	0.141440680154E-05
5	3	9.602639670321296e-07	-7.109321691000948e-06	-.451874314921E-06	-.214955260696E-06
5	4	1.308791587892802e-06	2.224704260862664e-06	-.295320219610E-06	0.497947374945E-07
5	5	1.016882178549291e-07	2.710312553574803e-06	0.174795471385E-06	-.669368182338E-06
10	10	4.162659655849492e-06	-4.606371682870996e-07	0.100439125877E-06	-.238750300838E-07
30	28	6.568305862916830e-07	-2.596529055286230e-07	-.584441578684E-08	-.809375913544E-08
30	29	-1.302540173511650e-07	-3.778797546680055e-07	0.389521213237E-08	0.187969991193E-08
30	30	1.201884301780586e-06	1.058755985849718e-06	0.257229425698E-08	0.848735816571E-08
32	30	1.244450610811446e-06	6.611575012158759e-07	-.690056829098E-08	0.132472373219E-08
35	34	1.738217205640891e-06	1.195386400870544e-06	-.798674881563E-09	0.276325910655E-08
50	49	3.837880963328032e-08	3.879891478301319e-07	0.215868304713E-08	-.475090336407E-08
50	50	-3.255351105146776e-07	1.119227112994072e-07	0.456424766124E-08	0.264773190543E-08

Table 4. Results from GOCE observations and tsvd calculations.

		tsvd		GOCE	
n	m	C	S	C	S
2	0	-1.470197649884266e-09	5.334668011946423e-22	-4.841649667904e-04	0
2	1	6.759276213896078e-13	-3.331482438287403e-11	-2.525700823662e-10	1.395515029905e-09
2	2	-1.370077052803747e-12	-5.305271081885416e-14	0.243936742466E-05	-.140024885157E-05
3	2	-6.471656119025104e-12	-2.463736347916412e-13	0.904802178874E-06	-.618981455772E-06
3	3	2.950619281576821e-14	-4.274029248718163e-13	0.721276848787E-06	0.141440680154E-05
5	3	2.068373751297736e-13	-3.661284931155739e-12	-.451874314921E-06	-.214955260696E-06
5	4	3.682932707326545e-13	2.720022557984094e-14	-.295320219610E-06	0.497947374945E-07
5	5	-2.336454032474866e-13	2.551163310617606e-12	0.174795471385E-06	-.669368182338E-06
10	10	-3.959498664028026e-13	-7.164501347578725e-14	0.100439125877E-06	-.238750300838E-07
30	28	1.536194410732852e-13	8.357047632649088e-14	-.584441578684E-08	-.809375913544E-08
30	29	-7.827779270823781e-14	1.409844732349580e-13	0.389521213237E-08	0.187969991193E-08
30	30	-1.712565002706274e-13	-1.012747246537633e-13	0.257229425698E-08	0.848735816571E-08
32	30	1.454265644978689e-13	8.588291048643461e-14	-.690056829098E-08	0.132472373219E-08
35	34	5.807440847463732e-14	3.987798475016890e-14	-.798674881563E-09	0.276325910655E-08
50	49	-2.031351418064919e-13	1.763888199780871e-13	0.215868304713E-08	-.475090336407E-08
50	50	-1.683076409847314e-13	-2.035208653455976e-13	0.456424766124E-08	0.264773190543E-08

Table 5. Results from GOCE observations and OSVRM calculations.

		OSVRM		GOCE	
n	m	C	S	C	S
2	0	-4.841653716980924e-04	0	-4.841649667904e-04	0
2	1	-1.869838784676788e-10	1.195284616735103e-09	-2.525700823662e-10	1.395515029905e-09
2	2	2.439143524328988e-06	-1.400166836708963e-06	0.243936742466E-05	-.140024885157E-05
3	2	9.046277663688434e-07	-6.190259435393191e-07	0.904802178874E-06	-.618981455772E-06
3	3	7.210726574739763e-07	1.414356273517268e-06	0.721276848787E-06	0.141440680154E-05
5	3	-4.519554076625794e-07	-2.148471966072914e-07	-.451874314921E-06	-.214955260696E-06
5	4	-2.953016466386062e-07	4.966588838441174e-08	-.295320219610E-06	0.497947374945E-07
5	5	1.749719880227612e-07	-6.693842777681447e-07	0.174795471385E-06	-.669368182338E-06
10	10	1.005386352334964e-07	-2.401484587209459e-08	0.100439125877E-06	-.238750300838E-07
30	28	-5.471172467689049e-09	-7.959978369346329e-09	-.584441578684E-08	-.809375913544E-08
30	29	4.159277718154803e-09	1.894907465430104e-09	0.389521213237E-08	0.187969991193E-08
30	30	2.647761935878369e-09	8.129944035816356e-09	0.257229425698E-08	0.848735816571E-08
32	30	-6.747869733339746e-09	1.393468736231077e-09	-.690056829098E-08	0.132472373219E-08
35	34	-1.216265171080703e-09	2.667190709351536e-09	-.798674881563E-09	0.276325910655E-08
50	49	2.270839003170000e-09	-4.597862628212736e-09	0.215868304713E-08	-.475090336407E-08
50	50	5.438245203826104e-09	1.480856365338055e-09	0.456424766124E-08	0.264773190543E-08

Table 6. RMSE between different methods of regularization.

n	Tikhonov vs. GOCE	OSVRM vs. GOCE	EGM vs. GOCE
1	1.728962688525945e-04	1.449863968187492e-14	2.662976961035196e-10
2	1.535721226393333e-04	2.878016160045136e-14	3.159063134497055e-10
10	9.364585630275397e-05	3.767275199513147e-13	7.976034857584555e-10
11	9.015072974311100e-05	4.607725786820048e-13	8.890537083131126e-10
20	6.994759127416688e-05	1.617659257007359e-12	1.581779723306636e-09
25	6.330089787551022e-05	2.531216944802798e-12	2.003336075722611e-09
30	5.824791973523789e-05	3.474765952840413e-12	2.630160153609919e-09
35	5.424253906182881e-05	4.430400087191455e-12	3.110848253514074e-09
40	5.096650693250293e-05	5.526221021849382e-12	3.454630169033278e-09
45	4.823005712334312e-05	6.730811224716724e-12	3.774912925035239e-09
49	4.635811940248126e-05	7.583174470172289e-12	3.999108722938432e-09

the RMSE of the results of OSVRM method – that use a vector instead of scalar as regularization parameter – isin the order of 10^{-13} which is much better from two other methods.

Briefly, the coefficients obtained from the OSVRM method are much closer to the GOCE coefficients concerning the results of other methods. In other words, the results show a significant improvement in the accuracy of the estimated coefficients.

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